ON IDENTIFICATION AND THE GEOMETRY OF THE SPACE OF LINEAR SYSTEMS

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1. INTRODUCTION AND MOTIVATION

Let

$$\dot{x} = Fx + Gu \qquad\qquad x_{t+1} = Fx_t + Gu_t$$

(1.1)

$$y = Hx \qquad\qquad y_t = Hx_t$$

be a continuous time or discrete time linear dynamical system of state space dimension n, with m inputs and with p outputs. (So that $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$). Here the matrices F,G,H are supposed to be independant of time. We use $L_{m,n,p} = \mathbb{R}^{mn+np+n^2}$ to denote the space of all such systems, and we let $L^{co}_{m,n,p}$ (resp. $L^{cr}_{m,n,p}$, resp. $L^{co,cr}_{m,n,p}$) denote the open and dense subspaces of all completely observable (resp. completely reachable, resp. completely observable and completely reachable) systems. Base change in state space induces an action of GL_n, the group of real invertible n x n matrices on $L_{m,n,p}$, viz. $(F,G,H)^S = (SFS^{-1}, SG, HS^{-1})$, and two systems in $L_{m,n,p}$ which are related in this way are indistinguisable from the point of view of their input-output behaviour. Inversely, if (F,G,H), $(\bar{F},\bar{G},\bar{H})$ are two systems in $L_{m,n,p}$ with the same input-output behaviour and at least one of them is cr and co then they are GL_n-equivalent (i.e. there is an $S \in GL_n$ such that $(\bar{F},\bar{G},\bar{H}) = (F,G,H)^S$). This makes the space of orbits $M^{co,cr}_{m,n,p} = L^{co,cr}_{m,n,p}/GL_n$ important in identification of systems theory, essentially because the input-output data of a given black box give zero information concerning a basis for state space. More precisely suppose we have given a black-box which is to be modelled by means of a linear dynamical system. Then the input-output data give us (hopefully) a point of $M^{co,cr}_{m,n,p}$ (for some more remarks concerning this cf. below in 1.10). As more and more input-output data come in we find a sequence of points in $M^{co,cr}_{m,n,p}$ representing better and better n-dimensional linear dynamical system approximations of the given black box. If this sequence approaches a limit we have found the best linear dynamical system model (of dimension n) of our black box. We have then "identified" the black box. The same picture is relevant if we are dealing with a slowly time varying linear dynamical system. (In practice of course it is often desirable to have a concrete representation in terms of triples of matrices of our sequence of systems; this is where the matter of continuous canonical forms comes in). Unfortunately the space $M^{co,cr}_{m,n,p}$ is never compact; i.e. a sequence of points in $M^{co,cr}_{m,n,p}$ may fail to converge. There are holes in $M^{co,cr}_{m,n,p}$. To illustrate what kinds of holes there are we offer the following three 2-dimensional, 1 input-1 output examples.

1.2. Example.

$g_a = \binom{1}{1}$, $F_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $h_a = (a,0)$. The result of starting in $x_o = 0$ at time $t = 0$ with the input function $u(t)$ is then

$$y(t) = \int_o^t (1+t-\tau)ae^{t-\tau}u(\tau)d\tau \tag{1.3}$$

Taking e.g. $u(t) = 1$ for $0 \leq t \leq T$ and $u(t) = 0$ for $t > T$ we see that the family of systems $(F_a,g_a,h_a)_a$ does not have any reasonable limiting input-output behaviour as $a \to \infty$. Such a family can hardly represent a sequence of better and better approximations to any (physical or economical) black box.

1.4. Example.

$g_a = \binom{a}{1}$, $F_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $h_a = (a^{-1},0)$, $0 < a \in \mathbb{R}$. In this example the result of input $u(t)$, starting in $x_o = 0$ at $t = 0$, is the output

$$y(t) = \int_o^t h_a e^{(t-\tau)F_a} g_a u(\tau)d\tau = \int_o^t e^{t-\tau}u(\tau)d\tau + \int_o^t a^{-1}e^{t-\tau}(t-\tau)u(\tau)d\tau \tag{1.5}$$

We see that the limiting input/output behaviour of this family of systems as $a \to \infty$ is the same as that of the 1-dimensional system $g = 1$, $F = 1$, $h = 1$. This kind of hole is of course expected. Obviously a family of systems (g_a,F_a,h_a) may "suddenly" have zero-pole cancellation as $a \to \infty$. The example also illustrates that the family of systems itself $(g_a,F_a,h_a)_a$ may not converge to anything as $a \to \infty$, while the family of input-output operators

$$U_a\colon u(t) \mapsto y_a(t) = \int_o^t h_a e^{(t-\tau)F_a} g_a u(\tau)d\tau \tag{1.6}$$

does converge as $a \to \infty$ (In the pointwise, i.e. weak topology, sense that

$\lim_{a\to\infty} U_a(u(t))$ exists for each sufficiently nice $u(t)$). This type of phenomenon is of course expected if one takes quotients with respect to the action of a noncompact group.

1.7. Example. $g_a = \binom{1}{1}$, $F_a = \begin{pmatrix} -a & -a \\ 0 & -a \end{pmatrix}$, $h_a = (a^2,0)$, $a \in \mathbb{R}$. In this case the limit

$$\lim_{a\to\infty} y_a(t) = \lim_{a\to\infty} \int_0^t e^{-a(t-\tau)}(a^2-a^3(t-\tau))u(\tau)d\tau \tag{1.8}$$

does exist for all reasonable input functions $u(t)$. (E.g. continuously differentiable input functions). The limit operator is in fact the differentiation operator $D: u(t) \mapsto y(t) = \frac{du(t)}{dt}$. But this operator is not the input-output operator of any system of the form (1.1). E.g. because D is unbounded, while the input-output operators of systems of the form 1.1 are necessarily bounded.

1.9. The Example 1.7 also shows that an obvious first thing to try: "just add in some nice way the lower dimensional systems" will not be sufficient at least for continuous time systems. However, even for discrete time systems, where as we shall see, the phenomenon of example 1.7 cannot occur, "adding in the lower dimensional systems" is of doubtful utility. To see this we turn our attention to a second bit of motivation for studying possible compactifications of $M^{co,cr}_{m,n,p}$. This has to do with finding a point in $M^{co,cr}_{m,n,p}$ which approximates, in some to be specified sense, a given set of input-output data, a point which was skipped over somewhat lightly in the first paragraph of this introduction. Incidentally it is reasonable to try to limit one's attention to co and cr systems because only the co and cr part of a system is deducible from its input-output behaviour. Also the quotient $L_{m,n,p}/GL_n$ is not Hausdorff, while $L^{co,cr}_{m,n,p}/GL_n$ is a nice smooth manifold (cf. [1]), so that the abstract mathematics and the more physical interpretation agree rather well.

1.10. On finding best $\leq$ n-dimensional linear system approximations to given input-output data. To avoid a number of far from trivial extra difficulties which adhere to the continuous time case we here concentrate on discrete time systems. Suppose therefore that we have input-output data relating inputs $u(t)$, $t = 0,1, \ldots, T-1$ to outputs $y(t)$, $t = 1, \ldots, T$ and that, for various reasons, e.g. economy of data storage, we wish to model this relationship by means of a discrete time system (1.1). Here n is supposed to be small comparent to T. One straightforward way to approach this in the 1 input-1 output case is as follows. Every cr triple $(F,g,h) \in L_{1,n,1}$ is GL_n equivalent to one of the form

$$(1.11)\qquad g = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdot & \ddots & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & \cdot & \cdot & 0 & 1 \\ -a_o & \cdot & \cdot & \cdot & -a_{n-1} \end{pmatrix}, \quad h = (b_o, \ldots, b_{n-1})$$

This results in the following ARMA relationship between inputs and outputs

$$(1.12)\qquad y_{N+n} + a_{n-1}y_{N+n-1} + \cdots + a_1 y_{N+1} + a_o y_N = b_{n-1}u_{N+n-1} + \cdots + b_1 u_{N+1} + b_o u_N$$

for all $N \geq 0$, $N \leq T-n$. And, inversely, an ARMA model like (1.12) implies that the input-output relationship can be thought of as generated by an underlying discrete dynamical system (1.1) which is GL_n-equivalent to one with its matrices as in (1.11).

Our input-output data give a collection of vectors $d = (z_n, \ldots, z_o; v_{n-1}, \ldots, v_o) \in \mathbb{R}^{2n+1}$ and it remains to find that hyperplane defined by an equation of the form $Z_n + a_{n-1}Z_{n-1} + \ldots + a_o Z_o = b_{n-1}V_{n-1} + \ldots + b_1 V_1 + b_o V_o$ in $\mathbb{R}^{2n+1}$ which passes best through the collection of data points $\{d\}$. This seems straightforward enough and moreover an essentially linear procedure. There is only a small hint of trouble in that the hyperplane through zero such that e.g. the sums of the squares of the distances of the data points d to this hyperplane is minimal, may very well make only a very small angle with the hyperplane $Z_n = 0$. The problem of finding the best hyperplane is linear in the sense of projective geometry rather then affine geometry. A related difficulty is reflected by the fact that the natural limit of e.g. the family of ARMA schemes

$$(1.13)\qquad y_{N+2} + y_{N+1} + ay_N = au_{N+1} + u_N$$

as $a \to \infty$ is the relation $y_N = u_{N+1}$. But there is no discrete time linear dynamical system which can generate this relation, and it is also not true that the family of discrete time systems given by

$$(1.14)\qquad g = \binom{0}{1}, \quad F = \begin{pmatrix} 0 & 1 \\ -a & -1 \end{pmatrix}, \quad h = (1,a)$$

converges in input-output behaviour as $a \to \infty$. There is finally a hint of more possible trouble in the more inputs-more outputs case because in the one input-one output case the matrices of the form (1.11) induce a global continuous canonical form on $M^{co,cr}_{1,n,1}$ but in the case of m > 1 and p > 1 such global continuous canonical forms do not exist (and cannot exist) on all of $M^{cr,co}_{m,n,p}$, [1-4].

As it turns out the linearization carried out by (1.11) and (1.12) is rather more suspect that would be suggested by the remarks above. To see this we describe the situation as follows. There are natural bases of the space of all input functions and the space of all output functions, viz. the functions ε_i, $i = 0, \ldots, T-1$, $\varepsilon_i(t) = 0$ if $t \neq i$, $\varepsilon_i(i) = 1$ and η_i, $i = 1, \ldots, T$, $\eta_i(t) = 0$ if $t \neq i$, $\eta_i(i) = 1$.

Incidentally, in the discrete time, finite horizon case a different choice of basis does not essentially affect the picture to be described below. In the continuous time case, or in the discrete time case with infinite horizon the choice of bases in input- and output function space is much more consequential.

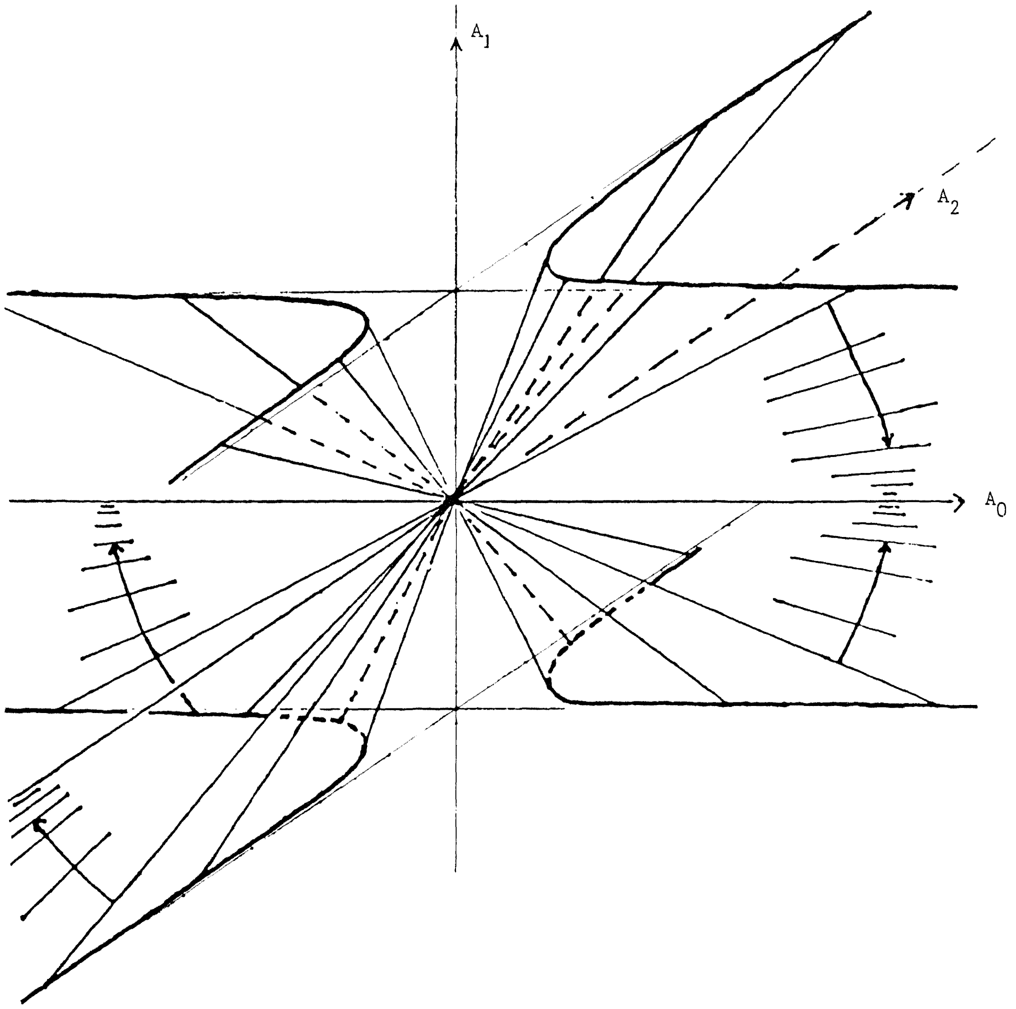
The space of all possible linear input-output relations (causal or not) is the space of all matrices

$$\begin{pmatrix} A_0 & \cdots & A_{0,T-1} \\ \vdots & & \vdots \\ A_{T-1,0} & \cdots & A_{T-1,T-1} \end{pmatrix}$$

(The causal input-output relations form a linear subspace). The space of input-output relations generated by a linear discrete time system of dimension $\leq$ n is an open dense subspace of the space of all matrices of Hankel form

$$H(A) = \begin{pmatrix} A_o & A_1 & \cdots & A_{T-1} \\ A_1 & & \cdot^{\cdot^{\cdot}} & \vdots \\ \vdots & \cdot^{\cdot^{\cdot}} & & \vdots \\ A_{T-1} & & \cdots & A_{2T-2} \end{pmatrix}$$

which moreover satisfy the condition rank $H(A) \leq n$. This is a highly nonlinear subspace, as is illistrated by the picture below which shows the closure of the subspace of input-output operators generated by a system of dimension ≤ 1 as a subspace of A_o, A_1, A_2 - space. The subspace is the cone with top in 0 through the hyperbola $A_1 = 1, A_0A_2 = 1$. The origin in the picture is the zero system and the points $A_o = 0$, $A_1 = 0$, $A_2 \neq 0$ are the points in the surface which are not realizable as ≤ 1 dimensional systems.



The nonlinearity of the picture is such as to suggest that it may will be impossible to linearize this surface without losing all a priori guarantees concerning the quality of our identification in terms of the noise in our data. This is indeed the case and to see this we calculate the sensitivity coefficients of the outputs y(1), y(2), y(3), .. with respect to the ARMA model parameters a_o, ..., a_{n-1}, b_o, ..., b_{n-1}. For simplicity we take n = 1. We write $a_o = -f$ and $b_o = h$. We then have of course

$$y(1) = hu(o),\ y(2) = hfu(0) + hu(1),\ y(3) = hf^2u(0) + hfu(1) + hu(2)$$

So that if, e.g., $u(1) = u(2) = 0$ and $u(0) = 1$, then the sensitivity coefficients of $y(1)$, $y(2)$, $y(3)$ with respect to the ARMA model parameters are respectively

$$\frac{\partial y}{\partial h} = (1,f,f^2),\quad \frac{\partial y}{\partial f} = (0,h,2hf)$$

which do not remain bounded independent of h and f. These sensitivity coefficients are especially had if both f and h are large. This fits with the remark made just above (1.13) above, because this corresponds to a hyperplane of best fit which is very close to the hyperplane $Z_n = 0$. On the other hand it is possible to divide the surface into a number of pieces and find local linearizations on each of these pieces such that the sensitivity coefficients calculated everywhere with respect to the appropriate local linearization do remain bounded. Indeed with respect to the coordinates A_o, A_1 we have $A_2 = A_o^{-1}A_1^2$ so that the sensitivity coefficients become

$$\frac{\partial y}{\partial A_o} = (1,0,-A_o^{-2}A_1^2),\ \frac{\partial y}{\partial A_1} = (0,1,2A_o^{-1}A_1)$$

and these are bounded by 2 in absolute value if $|A_o| \geq |A_1|$. On the other hand with respect to the coordinates A_1, A_2 we have $A_o = A_2^{-1}A_1^2$ so that the sensitivity coefficients become.

$$\frac{\partial y}{\partial A_1} = (2A_1A_2^{-1},1,0),\quad \frac{\partial y}{\partial A_2} = (-A_2^{-2}A_1^2,0,1)$$

and these are bounded by 2 in absolute value in the region where $|A_2| \geq |A_1|$. Now the surface has the equation $A_oA_2 = A_1^2$, so that for every point on the surface we must have $|A_o| \geq |A_1|$ or $|A_2| \geq |A_1|$ (or both). So we see that for this example two pieces suffice to find a piecewise linearization with uniformly bounded sensitivity coefficients. The picture incidentally suggests that to avoid trouble where both A_o and A_2 are small it would be good to introduce a third neighbourhood with coordinates A_1 and $\frac{1}{2}(A_o-A_2)$ in the intersection of the surface with, say, the solid cylinder $A_o^2 + A_2^2 \leq \frac{1}{2}$. The original coordinates h,f also work well in this region. It is perhaps also worth remarking that while the sensitivity coefficients $\frac{\partial y(n)}{\partial f}$, $\frac{\partial y(n)}{\partial h}$ get very rapidly worse if $f > 1$ and $n \to \infty$ this is much less so the case for the sensitivity coefficients $\frac{\partial y(n)}{\partial A_o}$, $\frac{\partial y(n)}{\partial A_1}$ and $\frac{\partial y(n)}{\partial A_1}$, $\frac{\partial y(n)}{\partial A_2}$ in their appropriate regions. Indeed in A_o, A_1 coordinates one has $A_n = A_o^{-n+1}A_1^n$ and in A_1, A_2 coordinates $A_n = A_1^{-n+2}A_2^{n-1}$ and the remark follows.

In the continuous time case we find instead of 1.12 a model

$$(1.15)\quad D^n y(t) + b_{n-1}D^{n-1}y(t) + \dots + b_o y(t) =$$

$$= a_{n-1}D^{n-1}u(t) + \dots + a_1 Du(t) + a_o u(t)$$

where D is again the differential operator. This model is already a priori more suspect than its discrete counterpart (1.12), simply because D is not a bounded operator.

1.16. The example suggests that it may be possible to construct the following sort of set up for identification procedures (discrete time case). There is a large open neighbourhood U of $\bar{\bar{M}}_{m,n,p}$, the closure in the space of all linear input-output relations of the space of those input-output matrices which are realizable by means of $\leq$ n dimensional linear systems. This neighbourhood U comes equipped with a finite covering U_i and coordinate maps $\phi_i : U_i \to \mathbb{R}^q$, $q = mpT^2$ such that $\phi_i(U_i \cap \bar{\bar{M}}_{m,n,p}) \subset \mathbb{R}^{mn+np} \subset \mathbb{R}^q$ (canonical embedding) and such that the Jacobian of ϕ_i is bounded on all of U_i for all i. The identification procedure would then roughly work as follows. Our input-output data give as a point in $\mathbb{R}^q$ the space of all linear input-output relations. If $x \notin U$, this input-output relation cannot be well approximated by a linear dynamical system of dimension $\leq$ n (and there should be an explicit number stating how badly the best approximation would still be). If $x \in U$, find an i such that $x \in U_i$. Apply ϕ_i to x and find the point $y \in \mathbb{R}^{mn+np} \subset \mathbb{R}^q$ closest to $\phi_i(x)$ (linear projection). Then take $\phi_i^{-1}(y)$ and this will be a good linear dynamical system approximation of the input-output operator x. The boundedness of the Jacobian of the ϕ_i guarantees that this procedure will have bounded sensitivity coefficients. In all this one can of course assume that x is already of Hankel form (if not first project on to the linear subspace of all input-output operators of Hankel form), so that the essential problem really is how curved $\bar{\bar{M}}_{m,n,p}$ lies in the space of all Hankel type matrices.

1.17. When can we expect that such a procedure can be constructed. Obviously this will be the case if we can find a suitable smooth Riemannian compactification of $M^{co,cr}_{m,n,p}$. Of course not every smooth compactification will do. The associated metric must fit with the topology on the space of the input-output operators belonging to the points of $M^{co,cr}_{m,n,p}$. The relevant topology on the space of operators appears to be the weak or pointwise-convergence topology. This is suggested by the results to be discussed below and also fits in well with (infinite dimensional) realization theory (Schwartz kernel theorem).

For instance the space of all cr systems of dimension n with one input and one output is $\mathbb{R}^{2n}$ and a nice smooth Riemannian compactification is the 2n-sphere S^{2n}, giving us also a nice smooth Riemannian compactification of $M_{1,n,1}$. Of course the same lower dimensional systems occur several times in the boundary of $M_{1,n,1}$ in S^{2n}; this, however, is not particularly bad for our purposes, and is a small price to pay for smoothness (and also appears to be unavoidable if one wants a smooth compactification). Much worse is that the one point compactification S^{2n} of $\mathbb{R}^{2n}$ brings systems very close together (in the Riemannian metric) which are very far from each other in input-output behaviour.

All this then is a second bit of motivation for studying (partial) compactifications of $M^{co,cr}_{m,n,p}$ which are system theoretically meaningful and for studying the degeneration possibilities of families of systems. Possibly, as is suggested by the results below, it is too much to hope for a total smooth Riemannian compactification. In that case one would try to find a smooth Riemannian partial compactification $\hat{M}_{m,n,p}$ which is system theoretically meaningful in the sense that a family of points in $\hat{M}_{m,n,p}$ converges to a point in $\hat{M}_{m,n,p}$ if the associated family of input-output operators converges in the weak topology (to some linear operator) and which has moreover the property that $\hat{M}_{m,n,p}$ is flat enough everywhere where it is not closed. This is precisely the situation one obtains if in the example above one adds to $M^{co,cr}_{1,1,1}$ the origin and the nonsystem points $A_o = 0$, $A_1 = 0$, $A_2 \neq 0$ and then resolves the singularity at the origin.

The remainder of this paper (sections 2-4) discusses some partial compactification results, these sections are essentially a somewhat revised version of the corresponding sections of [2].

2. DIFFERENTIAL OPERATORS OF ORDER $\leq$ n-1 AS LIMITS OF $L^{co,cr}_{1,n,1}$.

In this and the following section we consider continuous time systems only.

2.1. Definition. A differential operator of order n - 1 is (for the purposes of this paper) an input-output map of the form

$$y(t) = a_o u(t) + a_1 Du(t) + \ldots + a_{n-1} D^{n-1} u(t) \tag{2.2}$$

where the $a_o, \ldots, a_{n-1}$ are real constants and $a_{n-1} \neq 0$. The zero operator $u(t) \mapsto 0$ is, by definition, the unique differential operator of order -1. In this and the following section we shall always suppose that u(t) is as often continuously differentiable as is necessary.

2.3. <u>Theorem</u>. Let L be a differential operator of order $\leq n-1$. Then there exists a family of (continuous time) linear dynamical systems $(F_a, g_a, h_a)_a \subset L^{co,cr}_{1,n,1}$ such that (F_a, g_a, h_a) converges in input-output behaviour to L as $a \to \infty$. Here this last phrase means that for every smooth input function u(t) of compact support

$$(2.4) \qquad \lim_{a\to\infty} \int_o^t h_a e^{(t-\tau)F_a} g_a u(\tau)d\tau = Lu(t)$$

uniformly in t on every bounded t-interval in $[0,\infty)$.

2.5. To prove theorem 2.3 we do first some preliminary exercises concerning differentiation, partial integration and determinants. The determinant exercise is the following. Let $k = \mathbb{N} \cup \{0,-1\}$ and let $n \in \mathbb{N}$. Let $B(n,k)$ be the $n \times n$ matrix with the binomial coefficient entries $B(n,k)_{i,j} = \binom{i+j+k}{i+1+k}$, $i,j = 1, \ldots, n$. Then $\det(B(n,k)) = 1$ for all n,k. The combined differentiation/partial integration exercise says that

$$(2.6) \quad \int_o^t e^{-a(t-\tau)} a^n (t-\tau)^m u(\tau)d\tau = (-1)^m m! \sum_{i=m+1}^{n} (-1)^{i+1} a^{n-i} \binom{i-1}{m} u^{(i-1-m)}(t) + O(a^{-1})$$

where $u^{(j)}(t)$ is short for $\frac{d^j u}{dt^j}(t) = D^j u(t)$.

2.7. <u>Proof of theorem 2.3</u>. Let $1 \leq m \leq n$ and consider the following family of n-dimensional 1 input-1 output linear dynamical systems.

$$(2.8) \qquad g_a = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ a^m \end{pmatrix}, \quad F_a = \begin{pmatrix} -a & a & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & a \\ 0 & \cdots & \cdots & 0 & -a \end{pmatrix}, \quad h_a = (0,\ldots,0,b_m,\ldots,b_1)$$

where the $b_1, \ldots, b_m$ are still to be determined real numbers independant of the parameter a. Now sF_a is the sum of the diagonal matrix $-saI_n$ and the matrix with superdiagonal elements sa and zero's elsewhere. These matrices commute making it easy to write down e^{sF_a} explicitly and using this and (2.6) one finds without difficulty that

$$(2.9) \quad \int_o^t h_a e^{(t-\tau)F_a} g_a u(\tau)d\tau = \sum_{\ell=o}^{m-1} (-1)^{m-\ +1} a^{\ell} \Big(\sum_{i=1}^{m} b_i \binom{m+i-\ell-1}{i}\Big) u^{(m-\ell-1)}(t) + O(a^{-1})$$

Using the determinant result of 2.5 above it follows that we can choose $b_1, \ldots, b_m$ in such a way that

$$\int_o^t h_a e^{(t-\tau)F_a} g_a u(\tau)d\tau = bu^{(m-1)}(t) + O(a^{-1}) \tag{2.10}$$

where b is any pregiven real number. Now let L be any differential operator of order $\leq$ n-1, say $L = b_o + b_1D + \ldots + b_{n-1}D^{n-1}$. For each i = 0, ..., n-1 let $(F_a(i), g_a(i), h_a(i))$ be a family of dynamical systems such that (2.10) holds with $m - 1 = i$ and $b = b_i$. Now let (F'_a, g'_a, h'_a) be the n^2-dimensional system which is the direct sum of the n n-dimensional systems $(F_a(i), g_a(i), h_a(i))$. I.e.

$$g'_a = \begin{pmatrix} g_a(0) \\ \vdots \\ g_a(n-1) \end{pmatrix}, \quad F'_a = \begin{pmatrix} F_a(0) & & 0 \\ & \ddots & \\ 0 & & F_a(n-1) \end{pmatrix}, \quad h'_a = (h_a(0), \ldots, h_a(n-1)) \tag{2.11}$$

The transfer function of (F'_a, g'_a, h'_a) is then $T_a(s) = \sum_{i=o}^{n-1} h_a(i)(s-F_a(i))^{-1}g_a(i)$ and because $F_a(i)$ is the same matrix for all i it follows that the degree of the denominator of $T_a(s)$ can be taken to be $\leq$ n. By realization theory or decomposition theory, cf. [5], [6], it follows that there exists for all $a \in \mathbb{R}$ an n-dimensional system (F''_a, g''_a, h''_a) with transfer function $T_a(s)$, and the same input-output behaviour as (F'_a, g'_a, h'_a).

Finally because $L^{co,cr}_{1,n,1}$ is open and dense in $L_{1,n,1}$ we can find for all $a \in \mathbb{R}$ a cr and co system (F_a, g_a, h_a) such that

$$|h''_a e^{(t-\tau)F''_a} g''_a - h_a e^{(t-\tau)F_a} g_a| \leq \varepsilon_a |t-\tau| e^{|t-\tau| M_a}$$

where M_a is 1 plus the maximum of the absolute values of the entries of F''_a. Taking e.g. $\varepsilon_a = e^{-aM_a}$ we see that the families (F''_a, g''_a, h''_a) and (F_a, g_a, h_a) have the same limiting input-output behaviour. This concludes the proof of theorem 2.3

3. LIMITS OF TRANSFER FUNCTIONS

Let $(F,g,h) \in L^{co,cr}_{1,n,1}$. Its transfer function is $T(s) = h(s-F)^{-1}g$, which is a rational function of the form

$$T(s) = \frac{b_{n-1}s^{n-1} + \ldots + b_1 s + b_o}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_o} \tag{3.1}$$

such that numerator and denominator have no factors in common. The system (F,g,h) is up to GL_n equivalence uniquely determined by $T(s)$ so that we can and shall identify $M_{1,n,1}^{co,cr}$ with the space of all such rational functions (3.1). There is an obvious smooth compactification of this space of all rational functions, viz. $\mathbb{P}^{2n}$, real projective space of dimension 2n, which consists of all ratios $(x_o:\dots:x_{2n})$, $x_i \in \mathbb{R}$, such that at least one x_i is nonzero. We embed $M_{1,n,1}^{co,cr}$ in $\mathbb{P}^{2n}$ by mapping (F,g,h) to $(b_o:\dots:b_{n-1}:a_o,\dots,a_{n-1}:1)$, where the b_i and a_i are the transfer coefficients as in (3.1). The image of this mapping ψ is clearly open and dense.

Now let $\bar{M}_{1,n,1}$ be the subspace of $\mathbb{P}^{2n}$ consisting of those points $(x_o:\dots:x_{2n}) \in \mathbb{P}^{2n}$ for which at least one of the $x_n, \dots, x_{2n}$ is non-zero. To each $x \in \bar{M}_{1,n,1}$ we associate a (generalized) transfer function

$$\text{(3.2)} \qquad T_x(s) = \frac{x_{n-1}s^{n-1}+\dots+x_1s+x_o}{x_{2n}s^n+\dots+x_n} = c_{k-1}s^{k-1} + \dots + c_o + \frac{b_{n-k-1}s^{n-k-1}+\dots+b_o}{s^{n-k}+\dots+a_1s+a_o}$$

where $k = 2n - m$ if m is the index of the last coordinate of x which is nonzero. We write $L_x(s) = c_o + c_1s + \dots + c_{k-1}s^{k-1}$ and $T_x^r(s) = T_x(s) - L_x(s)$.

3.3. Lemma. Let $T_\alpha(s)$ be a family of transfer functions (3.1) of systems $(F_\alpha,g_\alpha,h_\alpha) \in L_{1,n,1}^{co,cr}$ indexed by a parameter α. Then $\lim_{\alpha\to\infty} T_\alpha(s)$ exists pointwise for infinitely many values of s iff (i) all limit points of the sequence $(x_\alpha)_\alpha$, $x_\alpha = \psi(F_\alpha,g_\alpha,h_\alpha)$, are in $\bar{M}_{1,n,1} \subset \mathbb{P}^{2n}$ and (ii) if x and x' are two limit points of this sequence then $T_x(s) = T_{x'}(s)$. Moreover if these conditions are fulfilled then $\lim_{\alpha\to\infty} T_\alpha(s) = T_x(s)$ for all limit points x of $(x_\alpha)_\alpha$.

The proof is elementary. Clearly if $(x_{\alpha'})_{\alpha'}$ is a subsequence of $(x_\alpha)_\alpha$ which converges to $x \in \bar{M}_{1,n,1}$ then $\lim_{\alpha'\to\infty} T_{\alpha'}(s) = T_x(s)$. Now suppose $(x_{\alpha'})_{\alpha'}$ is a subsequence which converges to some point in $\mathbb{P}^{2n} \setminus \bar{M}_{1,n,1}$, then $\lim_{\alpha'\to\infty} T_{\alpha'}(s) = \pm\infty$ for all but finitely many s. Finally if $(x_\alpha)_\alpha$ has all its limit points in $\bar{M}_{1,n,1}$ and there are limit points x, x' such that $T_x(s) \neq T_{x'}(s)$, then $\lim_{\alpha\to\infty} T_\alpha(s)$ cannot exist for infinitely many values of s because then we would have two unequal rational functions which are equal for infinitely many values of their argument.

3.4. Theorem. Let $x \in \overline{M}_{1,n,1}$ and let (F,g,h) be any $(n-k)$-dimensional system with transfer function equal to $T_x^r(s)$, and such that $\det(s-F) = s^{n-k} + x_m^{-1}x_{m-1} + \dots + x_m^{-1}x_{2n}$, where $m = 2n-k$ is the index of the last non zero coordinate of x. Then there exists a family of systems $(F_a,g_a,h_a) \subset L_{1,n,1}^{co,cr}$ such that

$$\lim_{a\to\infty} \int_0^t h_a e^{(t-\tau)F_a} g_a u(\tau)d\tau = L_x(D)u(t) + \int_0^t h e^{(t-\tau)F} g u(\tau)d\tau \tag{3.5}$$

and such that moreover

$$\lim_{a\to\infty} T_a(s) = T_x(s), \quad \lim_{a\to\infty} \psi(F_a,g_a,h_a) = x \tag{3.6}$$

where $T_a(s)$ is the transfer function of (F_a,g_a,h_a).

Proof. Let (F_a',g_a',h_a') be a family of k-dimensional systems in $L_{1,k,1}$ whose input-output behaviour converges to the differential operator $L_x(D)$. Let (F_a'',g_a'',h_a'') be the direct sum of (F_a',g_a',h_a') and (F,g,h). As in the proof of theorem 2.3 we can change the family (F_a'',g_a'',h_a'') to a family (F_a,g_a,h_a) of co and cr systems with the same limit input-output behaviour. Then (3.5) holds. The first part of (3.6) follows by taking $u(t)$ to be smooth of bounded support. Then the integrals and $L_x(D)u(t)$ in (3.5) are all Laplace transformable and the first part of (3.6) follows by the continuity of the Laplace transform (cf. [7], theorems 8.3.3 and 4.3.1). The second part of (3.6) follows from the first part together with the condition on $\det(s-F)$.

3.7. Theorem. Let (F_a,g_a,h_a) be a family of n-dimensional systems such that

$$\lim_{a\to\infty} \int_0^t h_a e^{(t-\tau)F_a} g_a u(\tau)d\tau$$

converges uniformly in t on bounded t intervals. Then there exists a $k \geq 0$, a differentail operator L of degree $\leq k-1$ and an $(n-k)$-dimensional system (F,g,h) such that

$$\lim_{a\to\infty} \int_0^t h_a e^{(t-\tau)F_a} g_a u(\tau)d\tau = Lu(t) + \int_0^t h e^{(t-\tau)F} g u(\tau)d\tau \tag{3.8}$$

Proof. By changing the (F_a,g_a,h_a) slightly if necessary (as in the proof of theorem 2.3) we can assume that $(F_a,g_a,h_a) \in L_{1,n,1}^{co,cr}$ for all a. Let $u(t)$ be a given smooth input function of bounded support and let $U(s)$ be its Laplace transform. The Laplace transform of the expresion under the limit sign in (3.8) is then $T_a(s)U(s)$, where $T_a(s)$ is the transfer function of (F_a,g_a,h_a). The continuity of the Laplace transform ([7], theorem 8.3.3) and lemma 3.3 above together then imply that there is an $x \in \overline{M}_{1,n,1}$ such that $\lim_{a\to\infty} T_a(s) = T_x(s)$. Take $L = L_x(D)$ and let (F,g,h) be any $(n-k)$-dimensional system with transfer function $T_x^r(s)$. Then the statement of the theorem follows because the Laplace transform is injective.

3.9. Theorems 3.4 and 3.7 together say that $\overline{M}_{1,n,1}$ is a maximal partial compactification in the sense that if a family of systems (F_a,g_a,h_a) converges in input-output behaviour then their associated points in $\overline{M}_{1,n,1}$ converge in $\overline{M}_{1,n,1}$, and inversely every point of $\overline{M}_{1,n,1}$ arises as a limit of a family $(x_a)_a$ which comes from a family of systems (F_a,g_a,h_a) which converges in input-output behaviour. It is not true, however, that a family (F_a,g_a,h_a) converges in input-output behaviour iff the sequence of associated points converges; cf. 3.10 below.

3.10. One cannot use realization theory directly to prove theorem 2.3. For instance the family of rational functions $(s-a)^{-1}a$ converges to -1 as $a \to \infty$ and -1 is the Laplace transform of the operator $u(t) \mapsto y(t) = -u(t)$. The transfer functions $(s-a)^{-1}a$ are realized by the systems $F = 1$, $g = 1$, $h = a$. But the limit $\lim_{a\to\infty} \int_0^t ae^{t-\tau}u(\tau)d\tau$ does not exist for almost all $u(t)$.

On the other hand the following is true. Let (F_a,g_a,h_a) be a family of systems with transfer functions $T_a(s)$. Suppose that there is a $c \in R$ such that $T_a(s)$ has no poles with real part $\geq c$ for all a. Then the limit of the $T_a(s)$ exists for $a \to \infty$ iff the family (F_a,g_a,h_a) converges in input-output behaviour. Half of this was proved in theorem 3.7 above. The other half is proved by using a continuity property of the inverse Laplace transform when applied to a converging sequence of rational functions with the extra property just mentioned.

This can be used to give another proof of theorem 2.3 as well as its obvious more input - more output generalization. The other theorems above generalize immediately to this case.

4. LIMITS OF DISCRETE TIME SYSTEMS

4.1. First let (F_a,g_a,h_a) be a family of co and cr continuous time systems of dimension n which converges in input-output behaviour. Let $A_i(a) = h_aF_a^ig_a$. Suppose in addition that for every i the $A_i(a)$ remain bounded. Then for every i there is a subsequence of $(A_i(a))_a$ which converges to some matrix A_i. Consider the block Hankel matrices

$$\mathcal{H}_{r,r}(a) = \begin{pmatrix} A_0(a) & \cdots & A_r(a) \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ A_r(a) & \cdots & A_{2r}(a) \end{pmatrix} \qquad \mathcal{H}_{r,r} = \begin{pmatrix} A_0 & \cdots & A_r \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ A_r & \cdots & A_{2r} \end{pmatrix}$$

By choosing the subsequences inductively we can see to it that a subsequence of $\mathcal{H}_{r,r}(a)$ converges to $\mathcal{H}_{r,r}$. It follows that $\mathrm{rank}(\mathcal{H}_{r,r}) \leq n$ for all r, which in turn (cf. [5], chapter 10) means that $A_0, A_1, A_2, \ldots$ is realizable by a $\leq n$ dimensional system. From this we see that the limit input-output behaviour of the family (F_a,g_a,h_a) is necessarily the input-output behaviour of a $\leq n$ dimensional system. I.e. the extra boundedness assuption on the $A_i(a)$ sees to it that the limit differential operator L occurring in (3.8) is always zero.

4.2. Now let (F_a, g_a, h_a) be a family of discrete time systems. The input-output operator of (F_a, g_a, h_a) is the matrix $(A_0(a) \mid A_1(a) \mid \ldots)$. Now assume that the (F_a, g_a, h_a) are n-dimensional and that the family converges in input-output behaviour. Then the $A_i(a)$ remain bounded for all i, and argueing exactly as in 4.1 above we find that the limit input-output behaviour is that of a linear discrete time system, possibly of lower dimension. In other words, in the discrete time time case a maximal partial compactification of $M^{co,cr}_{1,n,1}$ is the space $\hat{M}_{1,n,1}$ consisting of all $(x_0 : x_1 : x_2 : \ldots : x_{2n}) \in P^{2n}$ such that the polynomial part of the associated rational function, $L_x(s)$, is zero. That is, the smooth partial compactification $\hat{M}_{1,n,1}$ is obtained by adding in (several times) all lower dimensional systems and nothing else.

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